

MONOMIAL IDEALS WHOSE DEPTH FUNCTION HAS ANY GIVEN NUMBER OF STRICT LOCAL MAXIMA

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ABSTRACT. We construct monomial ideals with the property that their depth function has any given number of strict local maxima.

In recent years there have been several publications concerning the stable set of prime ideals of a monomial ideal, see for example [4],[6], [13] and [12]. It is known by Brodmann [2] that for any graded ideal I in the polynomial ring S (or any proper ideal I in a local ring) there exists an integer k_0 such that $\text{Ass}(I^k) = \text{Ass}(I^{k+1})$ for $k \geq k_0$. The smallest integer k_0 with this property is called the index of stability of I and $\text{Ass}(I^{k_0})$ is called the set of stable prime ideals of I . A prime ideal $P \in \bigcup_{k \geq 1} \text{Ass}(I^k)$ is said to be persistent with respect to I if whenever $P \in \text{Ass}(I^k)$ then $P \in \text{Ass}(I^{k+1})$, and the ideal I is said to satisfy the persistence property if all prime ideals $P \in \bigcup_{k \geq 1} \text{Ass}(I^k)$ are persistent. It is an open question (see [7] and [15, Question 3.28]) whether any squarefree monomial ideal satisfies the persistence property.

We call the numerical function $f(k) = \text{depth}(S/I^k)$ the depth function of I . It is easy to see that a monomial ideal I satisfies the persistence property if all monomial localizations of I have a non-increasing depth function. In view of the above mentioned open question it is natural to ask whether all squarefree monomial ideals have non-increasing depth functions. The situation for non-squarefree monomial ideals is completely different. Indeed, in [10, Theorem 4.1] it is shown that for any non-decreasing numerical function f , which is eventually constant, there exists a monomial ideal I such that $f(k) = \text{depth}(S/I^k)$ for all k . Note that a similar result for non-increasing depth functions is not known, even it is expected that all square-free monomial ideals have non-increasing depth functions. In general the depth function of a monomial ideal does not need to be monotone. Examples of monomial ideals with non-monotone depth function are given in [14, Example 4.18] and [10]. The question arises which numerical functions are depth functions of monomial ideals. Since $\text{depth}(S/I^k)$ is constant for all $k \gg 0$ (see [1]), any depth functions is eventually constant. So the most wild conjecture one could make is that any numerical function which is eventually constant is indeed the depth function of a monomial ideal. In support of this conjecture we show in our theorem that for any given number n there exists a monomial ideal whose depth function has precisely

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n strict local maxima. The price that we have to pay to obtain such examples is that the number of variables needed to define our monomial ideal with n strict local maxima is relatively large, namely $2n + 4$. For this class of examples the depth function is constant beyond the number of variables. In all other examples known to us, in particular those discussed in [10], this is also the case. Thus we are tempted to conjecture that for any monomial ideal I in a polynomial ring in n variables $\text{depth}(I^k)$ is constant for $k \geq n$.

In the following theorem we present the monomial ideals admitting a depth function as announced in the title of the paper.

Theorem 0.1. *Let $n \geq 0$ be an integer and $I \subset S = K[a, b, c, d, x_1, y_1, \dots, x_n, y_n]$ be the monomial ideal in the polynomial ring S with generators*

$$a^6, a^5b, ab^5, b^6, a^4b^4c, a^4b^4d, a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4x_ny_n^2, b^4x_n^2y_n.$$

Then

$$\text{depth}(S/I^k) = \begin{cases} 0, & \text{if } k \text{ is odd and } k \leq 2n + 1; \\ 1, & \text{if } k \text{ is even and } k \leq 2n; \\ 2, & \text{if } k > 2n + 1. \end{cases}$$

In particular, the depth function of this ideal has precisely n strict local maxima.

Proof. First of all, for each odd integer $k = 2t - 1$ with $t \leq n + 1$, we show that $\text{depth}(S/I^k) = 0$. For this purpose we find a monomial belonging to $(I^k : \mathfrak{m}) \setminus I^k$, where $\mathfrak{m} = (a, b, c, d, x_1, y_1, \dots, x_n, y_n)$. We claim that the monomial

$$u = a^4b^4(a^4x_1y_1^2)(b^4x_1^2y_1) \cdots (a^4x_{t-1}y_{t-1}^2)(b^4x_{t-1}^2y_{t-1})x_t y_t \cdots x_n y_n$$

satisfies $u \in (I^k : \mathfrak{m}) \setminus I^k$. Let

$$\begin{aligned} v_1 &= a^5b \cdot b^6(a^4x_{t-1}y_{t-1}^2) \prod_{i=1}^{t-2} (a^4x_iy_i^2)(b^4x_i^2y_i), \\ v_2 &= ab^5 \cdot a^6(b^4x_{t-1}^2y_{t-1}) \prod_{i=1}^{t-2} (a^4x_iy_i^2)(b^4x_i^2y_i), \\ v_3 &= a^4b^4c \prod_{i=1}^{t-1} (a^4x_iy_i^2)(b^4x_i^2y_i), \\ v_4 &= a^4b^4d \prod_{i=1}^{t-1} (a^4x_iy_i^2)(b^4x_i^2y_i), \end{aligned}$$

$$\begin{aligned}
v_{2\ell+3} &= a^6(b^4x_\ell^2y_\ell)^2 \prod_{i=1}^{\ell-1} (a^4x_iy_i^2)(b^4x_i^2y_i) \prod_{i=\ell+1}^{t-1} (a^4x_iy_i^2)(b^4x_i^2y_i), \quad 1 \leq \ell \leq t-1, \\
v_{2\ell+4} &= b^6(a^4x_\ell y_\ell^2)^2 \prod_{i=1}^{\ell-1} (a^4x_iy_i^2)(b^4x_i^2y_i) \prod_{i=\ell+1}^{t-1} (a^4x_iy_i^2)(b^4x_i^2y_i), \quad 1 \leq \ell \leq t-1, \\
v_{2\ell+3} &= (b^4x_\ell^2y_\ell) \prod_{i=1}^{t-1} (a^4x_iy_i^2)(b^4x_i^2y_i), \quad t \leq \ell \leq n, \\
v_{2\ell+4} &= (a^4x_\ell y_\ell^2) \prod_{i=1}^{t-1} (a^4x_iy_i^2)(b^4x_i^2y_i), \quad t \leq \ell \leq n.
\end{aligned}$$

Clearly $v_i \in I^k$ for $1 \leq i \leq 2n+4$. One easily see that

$$v_1 \mid a \cdot u, \quad v_2 \mid b \cdot u, \quad v_3 \mid c \cdot u, \quad v_4 \mid d \cdot u.$$

Since

$$a^4b^4(a^4x_\ell y_\ell^2)x_\ell = a^2(a^6(b^4x_\ell^2y_\ell))y_\ell, \quad a^4b^4(b^4x_\ell^2y_\ell)y_\ell = b^2(b^6(a^4x_\ell y_\ell^2))x_\ell,$$

it follows that

$$v_{2\ell+3} \mid x_\ell \cdot u, \quad v_{2\ell+4} \mid y_\ell \cdot u, \quad 1 \leq \ell \leq t-1.$$

Moreover,

$$v_{2\ell+3} \mid x_\ell \cdot u, \quad v_{2\ell+4} \mid y_\ell \cdot u, \quad t \leq \ell \leq n.$$

Hence $u \cdot \mathfrak{m} \subseteq I^k$. In other words, $u \in I^k : \mathfrak{m}$.

Now, we wish to prove that $u \notin I^k$. Since neither c nor d divides u , it is enough to show that $u \notin \bar{I}^k$, where

$$\bar{I} = (a^6, a^5b, ab^5, b^6, a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4x_ny_n^2, b^4x_n^2y_n).$$

Suppose that there exists a monomial $w = u_1 \cdots u_k \in \bar{I}^k$ with each $u_i \in G(\bar{I})$ such that w divides u . Since $\deg_{x_i}(u) = \deg_{y_i}(u) = 1$ for $i = t, \dots, n$, each u_i belongs to

$$\mathcal{M} = \{a^6, a^5b, ab^5, b^6, a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4x_{t-1}y_{t-1}^2, b^4x_{t-1}^2y_{t-1}\}.$$

Since $\deg_{x_i}(u) = \deg_{y_i}(u) = 3$ for $i = 1, \dots, t-1$, it follows that, for u_i and u_j belonging to

$$\mathcal{N} = \{a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4x_{t-1}y_{t-1}^2, b^4x_{t-1}^2y_{t-1}\}$$

with $i \neq j$, one has $u_i \neq u_j$. Since $|\mathcal{N}| = 2t-2 = k-1$, there exists $1 \leq j \leq k$ with $u_j \notin \mathcal{N}$. Let ρ denote the number of integers $1 \leq j \leq k$ with $u_j \notin \mathcal{N}$. Since w divides u , one has $\deg_a(w) \leq 4t$ and $\deg_b(w) \leq 4t$.

(a) Let $\rho = 1$. Since $|\mathcal{N}| = k-1$, each monomial belonging to \mathcal{N} divides w . Thus $\deg_a(w) = 4(t-1) + c$ and $\deg_b(w) = 4(t-1) + d$, where (c, d) belongs to $\{(0, 6), (1, 5), (5, 1), (6, 0)\}$. Hence one has either $\deg_a(w) > 4t$ or $\deg_b(w) > 4t$, a contradiction.

(b) Let $\rho = 2$. Then we may assume that $\deg_a(w) = 4(t-1) + c_1 + c_2$ and $\deg_b(w) = 4(t-2) + d_1 + d_2$, where each (c_i, d_i) belongs to $\{(0, 6), (1, 5), (5, 1), (6, 0)\}$. Again, one has either $\deg_a(w) > 4t$ or $\deg_b(w) > 4t$, a contradiction.

(c) Let $\rho = h$ with $h > 2$. Suppose that a^4 divides each of the monomials u_1, \dots, u_s , where $s \leq k - h$. Let $\deg_a(w) = 4s + c_1 + \dots + c_h$ and $\deg_b(w) = 4(k - h - s) + d_1 + \dots + d_h$, where $c_i + d_i = 6$ for each $1 \leq i \leq h$. Since

$$\deg_b(w) = 4(k - h - s) + (6 - c_1) + \dots + (6 - c_h) \leq 4t = 2(k + 1),$$

it follows that

$$\deg_a(w) = 4s + c_1 + \dots + c_h \geq 4(k - h) + 6h - 2(k + 1) = 2k + 2h - 2.$$

However, since $h > 2$, one has

$$2k + 2h - 2 > 2k + 4 - 2 = 2k + 2 = 2(k + 1) = 4t.$$

Thus $\deg_a(w) > 4t$, a contradiction.

The above discussions (a), (b) and (c) complete the proof of $u \notin I^k$. Hence u belongs to $(I^k : \mathfrak{m}) \setminus I^k$ and $\text{depth}(S/I^k) = 0$, as desired.

Now we are going to prove that $\text{depth}(S/I^k) \geq 1$ for any even number $k > 0$. For the proof we introduce the ideals $J = (a^6, a^5b, ab^5, b^6, a^4b^4c, a^4b^4d)$ and $L = (a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4x_ny_n^2, b^4x_n^2y_n)$. Then $I = J + L$, and hence

$$I^k = J^k + J^{k-1}L + \dots + J^2L^{k-2} + JL^{k-1} + L^k.$$

We first show that for $k \geq 2$, the factor module $(I^k : (c, d))/I^k$ is generated by the residue classes of the elements of set

$$(1) \quad \mathcal{S}_k = \{a^4b^4v_1 \cdots v_{k-1} : v_i \in G(L) \text{ and } v_i \neq v_j \text{ for } i \neq j\}.$$

Observe that the minimal set of generators of J^2 only consists of monomials in a and b . Therefore, the only monomials in I^k which are divisible by c or d are the generators of JL^{k-1} . It follows that the generators of $I^k : (c, d)$ which do not belong to I^k are the monomials of the form $a^4b^4v_1 \cdots v_{k-1}$ with $v_i \in G(L)$.

Suppose that $v_i = v_j$ for some $i \neq j$, say, $v_i = v_j = a^4x_\ell y_\ell^2$. We may assume that $i = 1$ and $j = 2$. Then

$$u = a^4b^4v_1 \cdots v_{k-1} = a^{12}b^4x_\ell^2y_\ell^4v_3 \cdots v_{k-1} = (a^{12})(b^4x_\ell^2y_\ell)v_3 \cdots v_{k-1}y_\ell^3.$$

Since $a^{12} \in J^2$ and since $b^4x_\ell^2y_\ell \in L$, we see that $u \in J^2L^{k-2} \subset I^k$. This proves (1).

For a monomial $u = a^4b^4v_1 \cdots v_{k-1} \in \mathcal{S}_k$, we set

$$Z_u = \{x_\ell : \deg_{x_\ell}(v_i) = 1 \text{ for some } i\} \cup \{y_\ell : \deg_{y_\ell}(v_i) = 1 \text{ for some } i\},$$

and

$$W_u = \bigcup_{x_\ell \notin \text{supp}(u)} \{x_\ell^2y_\ell, x_\ell y_\ell^2\}.$$

Note that $u + I^k$ is annihilated by a, b, c, d and all variables in Z_u and all monomials in W_u . Indeed, it is obvious that a, b, c, d and all monomials in W_u annihilate $u + I^k$. Now let $x_\ell \in Z_u$, we show that $ux_\ell \in I^k$. We can assume that $v_1 = a^4x_\ell y_\ell^2$. Hence $a^6(b^4x_\ell^2y_\ell)v_2 \cdots v_{k-1} \in I^k$ and $a^6(b^4x_\ell^2y_\ell)v_2 \cdots v_{k-1} | ux_\ell$, so $ux_\ell \in I^k$. Similarly for $y_s \in Z_u$, we show that $uy_s \in I^k$.

It follows from this observation that $(I^k : (c, d))/I^k$ is generated as K -module by the residue classes of monomials uvw where $u \in \mathcal{S}_k$, v is a monomial in the variables

x_i and y_j belonging to $V_u = \text{supp}(u) \setminus Z_u$ and w is a monomial in the variables not belonging to the support of u and not divisible by a monomial in W_u .

Fix $u = a^4 b^4 v_1 \cdots v_{k-1} \in \mathcal{S}_k$ and let $m = uvw$ be a generator of $(I^k : (c, d)) / I^k$ as described in the preceding paragraph. Then v is a monomial with $\deg_{x_i}(u) = \deg_{y_j}(u) = 2$ for each $x_i, y_j \in \text{supp}(v)$. After relabeling of the variables we may assume that

$$\text{supp}(u) = \{a, b, x_1, y_1, \dots, x_t, y_t\}.$$

Then

$$(2) \quad uv = a^4 b^4 \prod_{i=1}^r (a^4 x_i y_i^2) (b^4 x_i^2 y_i) \prod_{j=r+1}^s a^4 x_j y_j^{h_j} \prod_{\ell=s+1}^t b^4 x_\ell^{g_\ell} y_\ell$$

with $h_j \geq 2$ and $g_\ell \geq 2$, and $k - 1 = r + t$.

Claim (*): None of the monomials $m = uvw$ belongs to I^k .

For the proof of claim (*) we first observe

(#) If $w_1 \cdots w_s$ divides m with $w_1, \dots, w_s \in G(L)$, then $s \leq k - 1$ and after renumbering of the v_i we have $w_i = v_i$ for $i = 1, \dots, s$.

Indeed we may assume that $w_1 = a^4 x_j y_j^2$. It follows from (2) that $x_j y_j^2$ appears in one of v_i . Hence after renumbering we may assume that $w_1 = v_1$. Then $w_2 \cdots w_s$ divides m/v_1 . Induction on k completes the proof of (#).

Now in order to prove (*) we assume in the contrary that $m \in I^k$. Then there exist $w_i \in G(I)$ such that $w_1 \cdots w_k$ divides m . We may assume that $w_1, \dots, w_s \in G(L)$ and $w_{s+1}, \dots, w_k \in G(J)$. By (#) we may assume that $w_i = v_i$ for $i = 1, \dots, s$. Our next claim is the following:

(##) $s = k - 1$.

For the proof of (##) we consider the following two cases:

(i) Assume $s = k - 2$. Therefore, $w_{k-1} w_k$ divides $a^4 b^4 v_{k-1} v w$. However, since $w_{k-1} w_k \in G(J)$, we have $\deg_a(w_{k-1} w_k) > \deg_a(a^4 b^4 v_{k-1} v w)$ or $\deg_b(w_{k-1} w_k) > \deg_b(a^4 b^4 v_{k-1} v w)$, a contradiction.

(ii) Assume $s = k - h$ with $h > 2$. Hence $w_{k-h+1} \cdots w_k$ divides $a^4 b^4 v_{k-h+1} \cdots v_{k-1} v w$. Since $w_{k-h+1}, \dots, w_k \in G(J)$, it follows that

$$\deg_a(w_{k-h+1} \cdots w_k) + \deg_b(w_{k-h+1} \cdots w_k) \geq 6h.$$

On the other hand,

$$\deg_a(a^4 b^4 v_{k-h+1} \cdots v_{k-1} v w) + \deg_b(a^4 b^4 v_{k-h+1} \cdots v_{k-1} v w) = 4h + 4.$$

Now since $h > 2$, it follows that $4h + 4 < 6h$. This means that

$$\begin{aligned} \deg_a(a^4 b^4 v_{k-h+1} \cdots v_{k-1} v) + \deg_b(a^4 b^4 v_{k-h+1} \cdots v_{k-1} v) \\ < \deg_a(w_{k-h+1} \cdots w_k) + \deg_b(w_{k-h+1} \cdots w_k), \end{aligned}$$

a contradiction. This concludes the proof of (##).

Now as we know that $s = k - 1$, it follows that w_k divides $a^4 b^4 w$. This is a contradiction, since $w_k \in G(J)$. Thus the proof of $(*)$ is completed.

From claim $(*)$ it follow that $\text{depth}(S/I^k) > 0$ for even k . Indeed suppose that $\text{depth}(S/I^k) = 0$. Then $I^k : \mathfrak{m} \neq I^k$. Since $I^k : \mathfrak{m} \subset I^k : (c, d)$, it follows that there exists a monomial $m = uvw \in I^k : \mathfrak{m}$ of the form as described before. Now since k is even and $k \leq 2n$ and $v_i \neq v_j$ for $i \neq j$, the set $V_u \neq \emptyset$. It follows that $mv' \notin I^k$ for any $v' \in V_u$, a contradiction.

In the next step we show that $\text{depth}(S/I^k) \leq 1$ (and hence $\text{depth}(S/I^k) = 1$) for even k with $k \leq 2n$. Indeed, we claim that $P = (a, b, c, d, x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n)$ belongs to $\text{Ass}(I^k)$ for even k with $k \leq 2n$. Then, since

$$\text{depth}(S/I^k) \leq \min\{\dim(S/Q) : Q \in \text{Ass}(I^k)\}$$

(see [3, Proposition 1.2.13]), the required inequality follows.

To show this we note that $P \in \text{Ass}(I^k)$ if and only if $\text{depth}(S(P)/I(P)^k) = 0$, see for example [12, Lemma 2.3]. Here $S(P)$ is the polynomial ring in the variables which generate P and $I(P)$ is obtained from I by the substitution $y_n \mapsto 1$.

In our case $I(P)$ is generated by

$$a^6, a^5 b, ab^5, b^6, a^4 b^4 c, a^4 b^4 d, a^4 x_1 y_1^2, b^4 x_1^2 y_1, \dots, a^4 x_{n-1} y_{n-1}^2, b^4 x_{n-1}^2 y_{n-1}, a^4 x_n, b^4 x_n^2.$$

We claim that for $k = 2t$ with $t \leq n$ the monomial

$$u' = a^8 b^4 (a^4 x_1 y_1^2) (b^4 x_1^2 y_1) \cdots (a^4 x_{t-1} y_{t-1}^2) (b^4 x_{t-1}^2 y_{t-1}) x_t y_t \cdots x_{n-1} y_{n-1} x_n$$

satisfies $u' \in (I(P)^k : \mathfrak{m}(P)) \setminus I(P)^k$. This shows that $\text{depth}(S(P)/I(P)^k) = 0$. Let

$$v'_i = (a^4 x_n) v_i \text{ for } i = 1, \dots, 2n+2 \text{ and } v'_{2n+3} = a^6 (b^4 x_n^2) \prod_{i=1}^{t-1} (a^4 x_i y_i^2) (b^4 x_i^2 y_i)$$

where v_i is defined as in the first part of the proof. Clearly $v'_i \in I(P)^k$ for $1 \leq i \leq 2n+3$. one easily see that

$$v'_1 \mid a \cdot u', \quad v'_2 \mid b \cdot u', \quad v'_3 \mid c \cdot u', \quad v'_4 \mid d \cdot u'.$$

Moreover

$$v'_{2\ell+3} \mid x_\ell \cdot u', \quad v'_{2\ell+4} \mid y_\ell \cdot u, \quad 1 \leq \ell \leq n-1 \text{ and } v'_{2n+3} \mid x_n \cdot u'.$$

Hence $u' \in (I(P)^k : \mathfrak{m}(P))$.

With the same argument as given in the first part of the proof, one can easily see that $u' \notin I(P)^k$. Therefore $u' \in (I(P)^k : \mathfrak{m}(P)) \setminus I(P)^k$, so $\text{depth}(S(P)/I(P)^k) = 0$, as desired.

Finally we show that $\text{depth}(S/I^k) = 2$ for $k > 2n+1$. Since the only generators of I^k which are divisible by c are among the generators of JL^{k-1} we see that $I^k : (c) / I^k$ is generated by the set of monomials $\bigcup_{u \in \mathcal{S}_k} \{u, ud\}$. Since $k > 2n+1$, it follows that $\mathcal{S}_k = \emptyset$. Hence $I^k : (c) = I^k$ for $k > 2n+1$. Similarly, $I^k : (d) = I^k$ for $k > 2n+1$. It follows that c, d is a regular sequence on S/I^k for $k > 2n+1$. This implies that $\text{depth}(S/I^k) \geq 2$ for all $k > 2n+1$.

Let $\bar{S} = K[a, b, x_1, y_1, \dots, x_n, y_n]$ and

$$\bar{I} = (a^6, a^5b, ab^5, b^6, a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4x_ny_n^2, b^4x_n^2y_n) \subset \bar{S}.$$

Then $(S/I^k)/(c, d)(S/I^k) = \bar{S}/\bar{I}^k$.

We claim that $w = a^5b^{6k-6}x_1y_1x_2y_2 \cdots x_ny_n \in \bar{I}^k : \mathfrak{n} \setminus \bar{I}^k$ for $k \geq 2$, where \mathfrak{n} is the graded maximal ideal of \bar{S} . The claim implies that $\text{depth}(S/I^k/(c, d)S/I^k) = 0$ for all $k \geq 2$. In particular it follows that $\text{depth}(S/I^k) = 2$ for all $k > 2n + 1$, as desired.

To prove the claim we notice that aw is divisible by $(a^6)(b^6)^{k-1} \in \bar{I}^k$, and bw is divisible by $(a^5b)(b^6)^{k-1} \in \bar{I}^k$. Hence $aw, bw \in \bar{I}^k$.

Next observe that x_iw is divisible by $(a^5b)(b^6)^{k-2}(b^4x_i^2y_i) \in \bar{I}^k$ and y_iw is divisible by $(b^6)^{k-1}(a^4x_iy_i^2) \in \bar{I}^k$. This implies that $x_iw, y_iw \in \bar{I}^k$ for all i . Thus we have shown that $w \in \bar{I}^k : \mathfrak{n}$.

It remains to be shown that $w \notin \bar{I}^k$. Indeed, none of the generators of L divides w , because each of these generators has x_i -degree or y_i -degree 2. Therefore, if $w \in \bar{I}^k$, it follows that w is divisible by a monomial in a and b of degree $6k$. However, a^5b^{6k-6} has only degree $6k - 1$, a contradiction.

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